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RESEARCH ARTICLE

Some Basic Results in ν-Normed Spaces

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Abstract

In this paper, we introduce the notion of a directed preserving generator (d.p.g.) from $\mathbb{R}$ into $\mathbb{R}$. This d.p.g. can be used to construct new fields which generally have the same properties as R, except that some properties are affected by d.p.g. itself. With this new field, a ν-normed space will be formed. Some of the basic properties of this normed space are also discussed.

Keywords: field, normed space, Banach space, bounded operator.

1. Introduction

Sometimes we can discover new things by changing our point of view of something. Classical Banach spaces are defined as a complete normed space over the field $\mathbb{R}$ or $\mathbb{C}$. In this paper, we will use a new field called ν-non-Newtonian field ${}_\nu\mathbb{R}$ which has similar properties to $\mathbb{R}$. Using this ${}_\nu\mathbb{R}$ which will be constructed in this section, then we can define ν-non-Newtonian normed spaces over ν-non-Newtonian field ${}_\nu\mathbb{R}$. To simplify, we will just state the function needed. So, if the function is ν, the notion ν-normed spaces over a field ${}_\nu\mathbb{R}$ will be used if it is not ambiguous.

This change in point of view was started by Grossman and Katz by moving the field $\mathbb{R}$ by using a function called a generator. A function $\alpha : \mathbb{R} \to \mathbb{R}$ is called *a generator function* if it is an injective function [9]. Many studies have used generators with this definition in several fields such as calculus [1], α-fixed point theory [2, 3], and some special α-Banach spaces [4, 5, 6, 7, 8, 10, 11]. Unfortunately, the injective nature of this generator is not strong enough to guarantee that the field ${}_\alpha\mathbb{R}$ generated by α has similar properties as its counterpart $\mathbb{R}$. Therefore, we define the stronger generator as follows.

Definition 1. *A function* $\nu : \mathbb{R} \to \mathbb{R}$ *is called a* directed preserving generator *(g.d.p) if the function* ν *satisfies the following conditions:*
(i) one-one and continue,
(ii) for any $a, b \in \mathbb{R}$ *and* $a \leq b$, *we have* $\nu(a)\dot{\leq}\nu(b)$ *in* ${}_\nu\mathbb{R}$, *and*
(iii) for any $a, b \in \mathbb{R}$, *there exists* $\nu(c) \in {}_\nu\mathbb{R}$ *such that* $\nu(a)\dot{\leq}\nu(c)$ *and* $\nu(b)\dot{\leq}\nu(c)$ *in* ${}_\nu\mathbb{R}$.

The existence of $c \in \mathbb{R}$ in (*iii*) is just the implication of (*i*) and (*ii*). Using this new definition, the function $\alpha : \mathbb{R} \to \mathbb{R}$ which is defined as

$$\alpha(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{x}, & \text{otherwise} \end{cases}$$

is a generator that fails to be a g.d.p.

The continuity of g.d.p. ν will guarantee the existence of $\dot{0}$ and $\dot{1}$, where $\dot{0}$ and $\dot{1}$ respectively denote the addition identity and multiplication identity, i.e., $\nu(0) = \dot{0}$ and $\nu(1) = \dot{1}$. In the case $\nu = exp$, clearly $\dot{0} = 1$ and $\dot{1} = e$. The property (*ii*) of d.p.g. ensures that the order in ${}_\nu\mathbb{R}$ does not reverse the original order in $\mathbb{R}$, while (*iii*) ensures that there is always an element in ${}_\nu\mathbb{R}$ greater than $\nu(a), \nu(b) \in {}_\nu\mathbb{R}$, i.e., the elements of ${}_\nu\mathbb{R}$ (depend on g.d.p. ν) are going to $-\infty$ and $+\infty$.

Before going any further, for any $\dot{a}, \dot{b} \in {}_\nu\mathbb{R}$, the arithmetics applied in a set of scalars ${}_\nu\mathbb{R}$ is defined as follows

$$\dot{a}\dot{+}\dot{b} = \nu\left(\nu^{-1}(\dot{a}) + \nu^{-1}\left(\dot{b}\right)\right)$$

$$\dot{a}\dot{-}\dot{b} = \nu\left(\nu^{-1}(\dot{a}) - \nu^{-1}\left(\dot{b}\right)\right)$$

$$\dot{a}\dot{\times}\dot{b} = \nu\left(\nu^{-1}(\dot{a}) \times \nu^{-1}\left(\dot{b}\right)\right)$$

$$\dot{a}\dot{/}\dot{b} = \nu\left(\nu^{-1}(\dot{a}) / \nu^{-1}\left(\dot{b}\right)\right)$$

$$\dot{|}\dot{a}\dot{|} = \nu\left(\left|\nu^{-1}(\dot{a})\right|\right) = \begin{cases} \dot{a} & , \text{ if } \dot{a}\dot{>}\dot{0} \\ \dot{0} & , \text{ if } \dot{a}\dot{=}\dot{0} \\ \dot{0}\dot{-}\dot{a}, & \text{ if } \dot{a}\dot{<}\dot{0} \end{cases}$$

$$\dot{\sqrt{\dot{a}^{\dot{2}}}} = \dot{|}\dot{a}\dot{|}$$

$$\dot{a}^{\dot{p}} = \nu\left(\left[\nu^{-1}(\dot{a})\right]^{p}\right)$$

Using these arithmetics, for any $\dot{a}, \dot{b}, \dot{c} \in {}_\nu\mathbb{R}$ and a d.p.g. ν we have

$$\begin{aligned}
\dot{a}\dot{+}\left(\dot{b}\dot{+}\dot{c}\right) &= \nu\left(\nu^{-1}(\dot{a}) + \nu^{-1}\left(\dot{b}\dot{+}\dot{c}\right)\right) \\
&= \nu\left(\nu^{-1}(\dot{a}) + \nu^{-1}\left[\nu\left(\nu^{-1}\left(\dot{b}\right) + \nu^{-1}(\dot{c})\right)\right]\right) \\
&= \nu\left(\nu^{-1}(\dot{a}) + \nu^{-1}\left(\dot{b}\right) + \nu^{-1}(\dot{c})\right) \\
&= \nu\left(\nu^{-1}\left[\nu\left(\nu^{-1}(\dot{a}) + \nu^{-1}\left(\dot{b}\right)\right)\right] + \nu^{-1}(\dot{c})\right) \\
&= \nu\left(\nu^{-1}\left[\dot{a}\dot{+}\dot{b}\right] + \nu^{-1}(\dot{c})\right) \\
&= \left(\dot{a}\dot{+}\dot{b}\right)\dot{+}\dot{c},
\end{aligned}$$

and

$$\begin{aligned}
\dot{a}\dot{+}\dot{0} &= \nu\left(\nu^{-1}(\dot{a}) + \nu^{-1}(\dot{0})\right) \\
&= \nu\left(\nu^{-1}(\dot{a}) + 0\right) \\
&= \nu\left(\nu^{-1}(\dot{a})\right) = \dot{a}.
\end{aligned}$$

Similarly, it is easy to see that ${}_\nu\mathbb{R}$ is an abelian group under addition. Since

$$\dot{a}\dot{\times}\dot{1} = \nu\left(\nu^{-1}(\dot{a}) \times \nu^{-1}(\dot{1})\right) = \nu\left(\nu^{-1}(\dot{a})\right) = \dot{a}$$

and

$$
\begin{aligned}
\dot{a}\dot{\times}\left(\dot{b}\dot{+}\dot{c}\right) &= \nu\left(\nu^{-1}(\dot{a})\times\nu^{-1}\left(\dot{b}\dot{+}\dot{c}\right)\right)\\
&= \nu\left(\nu^{-1}(\dot{a})\times\nu^{-1}\left[\nu\left(\nu^{-1}\left(\dot{b}\right)+\nu^{-1}(\dot{c})\right)\right]\right)\\
&= \nu\left(\left[\nu^{-1}(\dot{a})\times\nu^{-1}\left(\dot{b}\right)\right]+\left[\nu^{-1}(\dot{a})\times\nu^{-1}(\dot{c})\right]\right)\\
&= \nu\left(\nu^{-1}\left[\nu\left(\nu^{-1}(\dot{a})\times\nu^{-1}\left(\dot{b}\right)\right)\right]+\nu^{-1}\left[\nu\left(\nu^{-1}(\dot{a})\times\nu^{-1}(\dot{c})\right)\right]\right)\\
&= \nu\left(\nu^{-1}\left[\dot{a}\dot{\times}\dot{b}\right]+\nu^{-1}\left[\dot{a}\dot{\times}\dot{c}\right]\right)\\
&= \left[\dot{a}\dot{\times}\dot{b}\right]\dot{+}\left[\dot{a}\dot{\times}\dot{c}\right],
\end{aligned}
$$

some routine calculations will show that ${}_{\nu}\mathbb{R}$ is a field. Indeed, the properties of d.p.g. assure that ${}_{\nu}\mathbb{R}$ is a complete field. Therefore, ${}_{\nu}\mathbb{R}$ can be used to form a new kind of normed space.

2. ν-Normed spaces

The previous section shows that there are new fields that can be formed from $\mathbb{R}$. If we can find an isomorphism from $\mathbb{R}$ onto ${}_{\nu}\mathbb{R}$, then we can form ν-normed spaces over ${}_{\nu}\mathbb{R}$.

Let ν and μ be d.p.g. and define an isomorphism $\iota : {}_{\mu}\mathbb{R}\to{}_{\nu}\mathbb{R}$ as $\iota(x)=\nu\left(\mu^{-1}(x)\right)$. Then for any $\dot{a},\dot{b}\in{}_{\mu}\mathbb{R}$

$$
\begin{aligned}
\iota\left(\dot{a}\dot{+}\dot{b}\right) &= \nu\left(\mu^{-1}\left(\dot{a}\dot{+}\dot{b}\right)\right)\\
&= \nu\left(\mu^{-1}\left[\mu\left(\mu^{-1}(\dot{a})+\mu^{-1}\left(\dot{b}\right)\right)\right]\right)\\
&= \nu\left(\mu^{-1}(\dot{a})+\mu^{-1}\left(\dot{b}\right)\right)\\
&= \nu\left(\mu^{-1}(\dot{a})\right)\ddot{+}\nu\left(\mu^{-1}(\dot{a})\right)\\
&= \iota(\dot{a})\ddot{+}\iota\left(\dot{b}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\iota\left(\dot{a}\dot{\times}\dot{b}\right) &= \nu\left(\mu^{-1}\left(\dot{a}\dot{\times}\dot{b}\right)\right)\\
&= \nu\left(\mu^{-1}\left[\mu\left(\mu^{-1}(\dot{a})\times\mu^{-1}\left(\dot{b}\right)\right)\right]\right)\\
&= \nu\left(\mu^{-1}(\dot{a})\times\mu^{-1}\left(\dot{b}\right)\right)\\
&= \nu\left(\mu^{-1}(\dot{a})\right)\ddot{\times}\nu\left(\mu^{-1}(\dot{a})\right)\\
&= \iota(\dot{a})\ddot{\times}\iota\left(\dot{b}\right).
\end{aligned}
$$

Since ${}_{\mu}\mathbb{R}$ and ${}_{\nu}\mathbb{R}$ are fields, we have $\iota\left(\dot{a}\dot{-}\dot{b}\right)=\iota(\dot{a})\ddot{-}\iota\left(\dot{b}\right)$ and $\iota\left(\dot{a}\dot{/}\dot{b}\right)=\iota(\dot{a})\ddot{/}\iota\left(\dot{b}\right)$ for any $\dot{b}\neq\dot{0}$. Considering the properties of d.p.g. ν and μ, we conclude that $\iota(x)$ is a field isomorphism. If μ is an identity mapping, i.e. $\mu(x)=x$, and ν be any d.p.g, then ι is an isomorphism from $\mathbb{R}$ onto ${}_{\nu}\mathbb{R}$.

Now we are ready to discuss ν-normed space X. Using the isomorphism defined above, for any $x, y \in X$ we have

$$\iota\left(\lambda\|x\| + \|y\|\right) = \lambda\dot{\times}\dot{\|}x\dot{\|}\dot{+}\dot{\|}y\dot{\|}.$$

Therefore, the followings are hold

(*i*) $\dot{\|}x\dot{\|}\dot{=}\dot{0}$ implies $\|x\| = \iota^{-1}\left(\dot{\|}x\dot{\|}\right) = \iota^{-1}\left(\dot{0}\right) = \iota^{-1}\left(\nu\left(0\right)\right) = 0$ and by the classical normed space rule we get $x = 0 \in X$. On the contrary, for $x = 0$, $\dot{\|}0\dot{\|} = \iota\left(\|0\|\right) = \dot{0}$,

(*ii*) $\dot{\|}\lambda\dot{\times}x\dot{\|}\dot{=}\iota\left(\|\lambda x\|\right)\dot{=}\iota\left(|\lambda|\,\|x\|\right)\dot{=}\dot{|}\lambda\dot{|}\dot{\times}\iota\left(\|x\|\right)\dot{=}\dot{|}\lambda\dot{|}\dot{\times}\dot{\|}x\dot{\|}$,

(*iii*) $\dot{\|}x\dot{+}y\dot{\|}\dot{=}\iota\left(\|x+y\|\right)\dot{\leq}\iota\left(\|x\|+\|y\|\right)\dot{=}\dot{\|}x\dot{\|}\dot{+}\dot{\|}y\dot{\|}$.

These facts give the following definition

Definition 2. *Let X be a vector space over the field ${}_{\nu}\mathbb{R}$. The function $\dot{\|}\cdot\dot{\|} : X \to {}_{\nu}\mathbb{R}^{+}$ is called a* ν-norm *on X if it satisfies*
(i) $\dot{\|}x\dot{\|}\dot{\geq}\dot{0}$ and $\dot{\|}x\dot{\|}\dot{=}\dot{0}$ if and only if $x = 0$,
(ii) $\dot{\|}\lambda\dot{\times}x\dot{\|}\dot{=}\dot{|}\lambda\dot{|}\dot{\times}\dot{\|}x\dot{\|}$,
(iii) $\dot{\|}x\dot{+}y\dot{\|}\dot{\leq}\dot{\|}x\dot{\|}\dot{+}\dot{\|}y\dot{\|}$
for all $x, y \in X$ and $\lambda \in {}_{\nu}\mathbb{R}$. The ordered pair $\left(X, \dot{\|}\cdot\dot{\|}\right)$ is called ν-normed space.

If the context being discussed is clear, then X will be preferred over $\left(X, \dot{\|}\cdot\dot{\|}\right)$. By using property (iii) in the definition or directly using the isomorphism ι, for all $x, y \in X$ we have

$$\dot{\|}x\dot{\|}\dot{=}\iota\left(\|x\|\right)\dot{\leq}\iota\left(\|x\dot{-}y\| + \|y\|\right)\dot{=}\dot{\|}x\dot{-}y\dot{\|}\dot{+}\dot{\|}y\dot{\|}.$$

Similarly

$$\dot{\|}y\dot{\|}\dot{=}\iota\left(\|y\|\right)\dot{\leq}\iota\left(\|y\dot{-}x\| + \|x\|\right)\dot{=}\dot{\|}x\dot{-}y\dot{\|}\dot{+}\dot{\|}x\dot{\|}.$$

Therefore

$$\begin{aligned}\dot{|}\,\dot{\|}x\dot{\|}\dot{-}\dot{\|}y\dot{\|}\,\dot{|} &\dot{=} max\left\{\dot{\|}x\dot{\|}\dot{-}\dot{\|}y\dot{\|},\ \dot{\|}y\dot{\|}\dot{-}\dot{\|}x\dot{\|}\right\}\\ &\dot{\leq} \dot{\|}x\dot{-}y\dot{\|}.\end{aligned}$$

The last inequality shows that the function $x \mapsto \dot{\|}x\dot{\|}$ is a ν-continuous function. Furthermore, for any $x, x_0, y, y_0 \in X$ and $\lambda, \lambda_0 \in {}_{\nu}\mathbb{R}$

$$\dot{\|}\left(x\dot{+}y\right)\dot{-}\left(x_0\dot{+}y_0\right)\dot{\|}\dot{\leq}\dot{\|}x\dot{-}x_0\dot{\|}\dot{+}\dot{\|}y\dot{-}y_0\dot{\|}$$

and

$$\begin{aligned}\dot{\|}\lambda x\dot{-}\lambda_0 x_0\dot{\|} &\dot{\leq} \dot{\|}\lambda x\dot{-}\lambda x_0\dot{\|}\dot{+}\dot{\|}\lambda x_0\dot{-}\lambda_0 x_0\dot{\|}\\ &\dot{=} \dot{|}\lambda\dot{|}\dot{\times}\dot{\|}x\dot{-}x_0\dot{\|}\dot{+}\dot{\|}x_0\dot{\|}\dot{|}\lambda\dot{-}\lambda_0\dot{|}.\end{aligned}$$

The continuity of a ν-norm function implies that the mappings $(x, y) \mapsto x\dot{+}y$ and $(\lambda, y) \mapsto \lambda\dot{\times}y$ are ν-continuous from X into X. After knowing the ν-continuity of these mappings, it is natural to define the convergence of sequences in ν-normed spaces. Note that while discussing some concepts related to sequences, since we use $\mathbb{N}$ as a directed set, it must be understood that the natural numbers being used are in the original order, not as outputs of d.p.g. ν.

Definition 3. *Let* (x_n) *be a sequence in* ν*-normed space* X*. The sequence* (x_n) *is said to be* ν-Cauchy *if for any* $\dot{\varepsilon}\dot{>}\dot{0}$*, there is* $N_{\dot{\varepsilon}} \in \mathbb{N}$ *such that* $\dot{\|}x_n\dot{-}x_m\dot{\|}\dot{<}\dot{\varepsilon}$ *whenever* $n, m\dot{\geq}N_{\dot{\varepsilon}}$*. The sequence* (x_n) ν-converges *to an element* $x \in X$ *if for any* $\dot{\varepsilon}\dot{>}\dot{0}$*, there is* $N_{\dot{\varepsilon}} \in \mathbb{N}$ *such that* $\dot{\|}x\dot{-}x_n\dot{\|}\dot{<}\dot{\varepsilon}$ *whenever* $n\dot{\geq}N_{\dot{\varepsilon}}$*. In this case,* x *is called* ν-limit point *of* (x_n) *and denoted by* ν*-lim* $(x_n) = x$*.*

The previous discussions show that we can use the ν-norm function to form the topology for a vector space X with $\mathbb{F} = {}_\nu\mathbb{R}$. The elements of this ν-norm topology are any neighborhoods of $x \in X$.

Definition 4. *Let* X *be* ν*-normed space and* $x \in X$*.* The closed ball *centered at* x *with a radii* $\dot{r}\dot{>}\dot{0}$ *is the set* $\left\{y \in X : \dot{\|}x\dot{-}y\dot{\|}\dot{\leq}\dot{r}\right\}$ *and* the open ball *is the set* $\left\{y \in X : \dot{\|}x\dot{-}y\dot{\|}\dot{<}\dot{r}\right\}$. The sphere *with a center at* x *and a radii* $\dot{r}$ *is the set* $\left\{y \in X : \dot{\|}x\dot{-}y\dot{\|}\dot{=}\dot{r}\right\}$*.*

If $x = 0$ and $\dot{r} = 1$, then they are called a unit closed ball, unit open ball, and unit sphere which are respectively denoted by ${}_\nu\overline{B}_X$, ${}_\nu B_X$, and ${}_\nu S_X$.

Definition 5. *A* ν*-normed space is* ν-Banach (complete) space *if any* ν*-Cauchy sequence in* X ν*-converges to an element* $x \in X$*.*

Çakmak and Başar [4] showed that some sequence spaces of scalars are Banach space with d.p.g. $\nu = exp$. As stated before, this result generally doesn't hold for any generator, unless satisfies d.p.g. conditions.

Let X be a vector space over a field ${}_\nu\mathbb{R}$, $A \subset X$, $x, y \in A$, and $\lambda \in {}_\nu\mathbb{R}$. If $\lambda \in \left(\dot{0}, \dot{1}\right)$ and $\lambda x\dot{+}\left(\dot{1}\dot{-}\lambda\right) y \in A$, then A is called a *convex set*. A is a *balanced set* if for any $\dot{|}\lambda\dot{|}\dot{\leq}\dot{1}$, we have $\lambda A \subset A$. If for any $z \in X$, there is $\lambda_z\dot{>}\dot{0}$ such that $z \in \beta A$ whenever $\beta\dot{>}\lambda_z$, then A is called *absorbing*.

Proposition 1. *Any ball in a* ν*-normed space is convex. Furthermore, if the ball is centered at the origin, then the ball is balanced and absorbing.*

Proof. Let ${}_\nu B_{\dot{r}}(x)$ be any ball centered at x with a radius $\dot{r}$. Take any $y, z \in {}_\nu B_{\dot{r}}(x)$ and $\dot{0}\dot{<}\lambda\dot{<}\dot{1}$, then

$$\begin{aligned}
\dot{\|}\lambda y\dot{+}\left(\dot{1}\dot{-}\lambda\right) z\dot{-}x\dot{\|} &\dot{=} \dot{\|}\lambda y\dot{+}\left(\dot{1}\dot{-}\lambda\right) z\dot{-}\lambda x\dot{-}\left(\dot{1}\dot{-}\lambda\right) x\dot{\|} \\
&\dot{=} \dot{\|}\lambda\left(y\dot{-}x\right)\dot{+}\left(\dot{1}\dot{-}\lambda\right)\left(z\dot{-}x\right)\dot{\|} \\
&\dot{\leq} \lambda\dot{\|}\left(y\dot{-}x\right)\dot{\|}\dot{+}\left(\dot{1}\dot{-}\lambda\right)\dot{\|}\left(z\dot{-}x\right)\dot{\|} \\
&\dot{\leq} \lambda\dot{r}\dot{+}\left(\dot{1}\dot{-}\lambda\right)\dot{r}.
\end{aligned}$$

Therefore, $\lambda y\dot{+}\left(\dot{1}\dot{-}\lambda\right) z \in {}_\nu B_{\dot{r}}(x)$.
Now, take a ball ${}_\nu B_{\dot{r}}(0)$ and $\dot{|}\lambda\dot{|}\dot{\leq}\dot{1}$. Then, for any $y \in {}_\nu B_{\dot{r}}(0)$, we get $\dot{\|}\lambda y\dot{\|}\dot{\leq}\dot{\|}y\dot{\|}$ and hence $\lambda y \in {}_\nu B_{\dot{r}}(0)$. This shows that ${}_\nu B_{\dot{r}}(0)$ is a balanced set. Indeed, for any $x \in X$ and $\beta\dot{>}\frac{\dot{\|}x\dot{\|}}{\dot{r}}$, then $x \in \beta\,{}_\nu B_{\dot{r}}(0)$. Thus ${}_\nu B_{\dot{r}}(0)$ is absorbing. □

Let A be any subset of a ν-normed space X. The closure of A, denoted by $\overline{A}$, is an intersection of all closed sets in X containing A. It is well known that $\overline{A}$ contains all ν-limit points. Since for any $x, y \in X$ and $x \dot{\neq} y$, we can find $\dot{r}\dot{>}\dot{0}$ such that ${}_\nu B_{\dot{r}}(x) \cap {}_\nu B_{\dot{r}}(y) = \emptyset$ which shows that ν-norm topology is Hausdorff, and the ν-limit point is unique. The interior of A is the union of all open subsets of A, i.e., the largest open set containing A and denote by A°.

Proposition 2. *Let* A *be any convex subset of a* ν*-normed space* X*. Then* $\overline{A}$ *and* A° *are convex sets.*

Proof. Let $x, y \in \overline{A}$, $x_n \to x$ and $y_n \to y$. Then, for any $\lambda \in (\dot{0}, \dot{1})$

$$\dot{\|}\lambda x \dot{+} (\dot{1} \dot{-} \lambda)\, y \dot{-} \lambda x_n \dot{+} (\dot{1} \dot{-} \lambda)\, y_n \| \dot{\leq} \lambda \dot{\|} x \dot{-} x_n \| \dot{+} (\dot{1} \dot{-} \lambda)\, \dot{\|} y \dot{-} y_n \|$$

and hence $\lambda x_n \dot{+} (\dot{1} \dot{-} \lambda)\, y_n \to \lambda y \dot{+} (\dot{1} \dot{-} \lambda)\, x$. Therefore, $\lambda y \dot{+} (\dot{1} \dot{-} \lambda)\, x \in \overline{A}$ which shows that $\overline{A}$ is a convex set.

Now, let $x, y \in A^{\circ}$. Then, for any $\lambda \in (\dot{0}, \dot{1})$

$$\lambda x \dot{+} (\dot{1} \dot{-} \lambda)\, y \in \lambda A^{\circ} \dot{+} (\dot{1} \dot{-} \lambda)\, A^{\circ} \subset A.$$

Since A is a convex set, it follows that A° is convex. □

3. Bounded operators

This section will discuss the mappings between ν-normed space X and Y. It is well known that a mapping $T : X \to Y$ is linear if $T(x \dot{+} y) = Tx \dot{+} Ty$ and $T(\lambda x) = \lambda T(x)$ for any $x, y \in X$ and $\lambda \in {}_{\nu}\mathbb{R}$. From elementary calculus, the mapping $f : X \to Y$ is said to be continuous if, for an arbitrary $\varepsilon > 0$, there is $\delta = \delta_{\varepsilon} > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Since we can form a ν-norm topology for ν-normed spaces X and Y, we can define the continuous linear operator as in the following definition.

Definition 6. *Let X and Y be ν-normed spaces and $T : X \to Y$ be a linear operator. T is a* continuous operator *if it is continuous between X and Y, where X and Y are considered topological spaces under ν-norm topology.*

Note that a subset A of a ν-normed space X is *bounded* if $\dot{\|} x \| \dot{\leq} M$ for any $x \in A$ and $M \dot{<} \infty$, i.e. $A \subset {}_{\nu}B_M(0)$ for a large M.

Proposition 3. *Let X and Y be ν-normed spaces and $T : X \to Y$ be a linear operator. The following conditions are equivalent:*
(i) T is continuous,
(ii) T is continuous at 0,
(iii) T is uniformly continuous,
(iv) There is a constant $M \dot{>} \dot{0}$ such that

$$\dot{\|} Tx \| \dot{\leq} M \dot{\|} x \|$$

for all $x \in X$.
(v) $T({}_{\nu}B_X)$ is a bounded set in Y.

Proof. $(i) \to (ii)$ is obvious.
$(ii) \to (i)$. Let (x_n) be any sequence ν-converges to $x \in X$. Since T continuous at 0,

$$\dot{\|} Tx \dot{-} Tx_n \| \dot{=} \dot{\|} T(x \dot{-} x_n) \| \to \dot{0},$$

or $Tx_n \to Tx$. Thus, T is a continuous operator.
$(ii) \to (iii)$. The linearity and continuity of T at 0 imply that for any $\dot{\varepsilon} \dot{>} \dot{0}$, there is $\dot{\delta} \dot{=} \dot{\delta}_{\dot{\varepsilon}}\, \dot{\varepsilon} \dot{>} \dot{0}$ such that $\dot{\|} Tx \| \dot{<} \dot{\varepsilon}$ whenever $\dot{\|} x \| \dot{<} \dot{\delta}$. Since X is a ν-normed space, there is $x_1, x_2 \in X$ such that $x_1 = x_2$. Thus, $\dot{\|} Tx_1 \dot{-} Tx_2 \| \dot{<} \dot{\varepsilon}$ whenever $\dot{\|} x_1 \dot{-} x_2 \| \dot{<} \dot{\delta}$. Since x is arbitrary and $\dot{\delta}$ just depend on $\dot{\varepsilon}$, T is a uniformly continuous operator.
$(iii) \to (i)$ is obvious.

$(iii) \to (iv)$. $\dot{\|}x\dot{\|}\dot{<}\dot{\delta}$ implies $\dot{\|}Tx\dot{\|}\dot{<}\dot{\varepsilon}$. For any $x \in X$, put $y = \frac{\dot{\delta}}{2\dot{\|}x\dot{\|}}x$. Then $\dot{\|}y\dot{\|}\dot{<}\dot{\delta}$ and hence $\dot{\|}Ty\dot{\|}\dot{<}\dot{\varepsilon}$. The linearity of T will give

$$\dot{\|}Tx\dot{\|}\dot{=}\frac{2\dot{\|}x\dot{\|}}{\dot{\delta}}\dot{\|}Ty\dot{\|}\dot{<}\frac{2\dot{\varepsilon}}{\dot{\delta}}\dot{\|}x\dot{\|}.$$

Set $M\dot{=}\frac{2\dot{\varepsilon}}{\dot{\delta}}$ to complete the proof.

$(iv) \to (iii)$. Taking a sequence (x_n) that converges to 0, then $\dot{\|}x_n\dot{\|} \to \dot{0}$ and hence $\dot{\|}Tx_n\dot{\|} \to \dot{0}$.

$(iv) \to (v)$. The equivalence can be done by taking $x \in {}_{\nu}B_X$.

$(v) \to (iv)$. Let $T({}_{\nu}B_X)$ be a bounded set, i.e., there is $M\dot{>}\dot{0}$ such that $\dot{\|}T({}_{\nu}B_X)\dot{\|}\dot{<}\dot{M}$. Put $y = \frac{x}{\dot{\|}x_n\dot{\|}} \in {}_{\nu}B_X$, then

$$\dot{\|}Tx\dot{\|}\dot{=}\dot{\|}x\dot{\|}\dot{\|}Ty\dot{\|}\dot{\leq}M\dot{\|}x\dot{\|}$$

and the proof is complete. □

Proposition 3 says that the notion of continuity and boundedness of linear operator T are interchangeable. Therefore, the following definition is equivalent to Definition 6.

Definition 7. *Let X and Y be ν-normed spaces and $T : X \to Y$ be a linear operator. T is* bounded *if $T(A)$ is bounded for any bounded set $A \subset X$. $B(X, Y)$ stands for a collection of all bounded linear operators from X into Y.*

The equivalence of (iv) and (v) in Proposition 3 gives a term called *operator norm* $\dot{\|}T\dot{\|}$ which is defined as

$$\dot{\|}T\dot{\|} = \sup\left\{\dot{\|}Tx\dot{\|} : x \in {}_{\nu}B_X\right\}.$$

Indeed, the number $\dot{\|}T\dot{\|}$ is the smallest number that satisfies (iv). To see this, assume the contrary, that is $\dot{0}\dot{\leq}M\dot{<}\dot{\|}T\dot{\|}$. Then, for any $x \in {}_{\nu}B_X$

$$\dot{\|}T\dot{\|} = \sup\left\{\dot{\|}Tx\dot{\|} : x \in {}_{\nu}B_X\right\}\dot{>}M\dot{\geq}M\dot{\|}x\dot{\|}$$

which contradicts the boundedness of T.

Theorem 1. *Let X be a ν-normed space and Y be a ν-Banach space. Then $B(X, Y)$ is a ν-Banach space.*

Proof. It is easy to see that $B(X, Y)$ is a ν-normed space under the operator norm. Let (T_n) be a ν-Cauchy sequence in $B(X, Y)$. Then, for each $x \in X$

$$\dot{\|}T_nx\dot{-}T_mx\dot{\|}\dot{\leq}\dot{\|}T_n\dot{-}T_m\dot{\|}\dot{\|}x\dot{\|}.$$

Since Y is a ν-Banach space, T_nx converges to some element $y \in Y$. Let $Tx\dot{=}y\dot{=}\nu\text{-}\lim T_nx$. Then, for any $x \in {}_{\nu}B_X$

$$\dot{\|}Tx\dot{-}T_mx\dot{\|}\dot{\leq}\dot{\|}Tx\dot{-}T_nx\dot{\|}\dot{+}\dot{\|}T_nx\dot{-}T_mx\dot{\|}.$$

Taking the supremum of both sides gives $\dot{\|}T\dot{-}T_m\dot{\|} \to \dot{0}$. Clearly, $T \in B(X, Y)$ and the proof is complete. □

Concluding Remark Since $\dot{\|}\cdot\dot{\|} \in {}_{\nu}\mathbb{R}$ and $\|\cdot\| \in \mathbb{R}$, by deploying the isomorphism ι, all the results in this paper are also true for classical normed spaces.

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References
[1] A. E. Bashirov, E. M. Kurpinar, and A. Özyapıcı, Multiplicative calculus and its applications, *J. Math. Anal. Appl.*, **337** (2008), 36–48.

[2] D. Binbaşıoğlu, S. Demiriz, and D. Türkoğlu, Fixed points of non-Newtonian contraction mappings on non-Newtonian metric spaces, *J. Fixed Point Theory Appl.*, **18** (2016), 213–224.

[3] D. Binbaşıoğlu, On fixed point results for generalized contractıons in non-Newtonian metric spaces, *Cumhuriyet Sci. J.*, **43** (2022), 289–293.

[4] A. F. Çakmak and F. Başar, Some new results on sequence spaces with respect to non-Newtonian calculus, *J. Inequal. Appl.*, **2012**, Article ID 932734, 12 pages.

[5] A. F. Çakmak and F. Başar, Certain spaces of functions over the field of non-Newtonian complex numbers, *Abstr. Appl. Anal.*, **2014**, Article ID 236124, 12 pages.

[6] N. Değirmen and B. Sağır, Some new notes on the bicomplex sequence spaces $l_p(\mathbb{BC})$, *J. Fract. Calc. Appl.,* **13** (2022), 66–76.

[7] N. Değirmen and B. Sağır, Some geometric properties of bicomplex sequence spaces $l_p(\mathbb{BC})$, *Konuralp J. Math.,* **10** (2022), 44–49.

[8] N. Değirmen and B. Sağır, Some fundamental properties of Banach space $l_p(\mathbb{BC}(N))$ with the $*$–Norm $\dot{\|} \cdot \dot{\|}_{2,l_p(\mathbb{BC}(N))}$, *Trans. A. Razmadze Math. Inst.,* **176** (2022), 183–195.

[9] M. Grossman and R. Katz, *Non-Newtonian Calculus*, (Lee Press, Pigeon Cove, Mass, 1872).

[10] N. Güngör, Some geometric properties of the non-Newtonian sequence $l_p(N)$, *Math. Slovaca.*, **70** (2020), 689–696.

[11] N. Sager and B. Sağır, On completeness of some bicomplex sequence spaces, *Palestine J. Math.,* **9** (2020), 891–902.