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RESEARCH ARTICLE

Multiplication of Matrices

Hasan Keleş*

Karadeniz Technical University, Department of Mathematics, Ortahisar, Trabzon 61080, Türkiye
*Corresponding author. Email: hasankeles@ktu.edu.tr

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Abstract

This study is about multiplication of matrices. Multiplication of real numbers, which can be written along a line, is also two way. Here, the direction is not an influential factor even when the elements are switched. For example 3.2 = 6 and 2.3 = 6. In matrices this makes left and right multiplication is mandatory. Left multiplication is already defined. This is multiplication in known matrices. Left multiplication is used in the studies since the definition of this operation until today. The most insurmountable situation here is that matrices do not commutative Property according to this operation. The left product is taken into account, when AB is written. Here the matrix A is made to be effective. This left product is denoted by AB. The right definition in this study is denoted by $\underleftarrow{AB}$. This operation multiplication is seen to be compatible with the left multiplication. The commutativity property in matrices is reinvestigated with this approach. The relation between the right multiplication and the Cracovian Product is given by J. Koci´nski (2004).

Keywords: Product, Multiplication of matrices, Matrix theory, Right product of matrices, Cracovian product.

1. Introduction

In 1850, Cayley is the first author to define the anchoring operation in matrices [1, 5]. The Cracovian Product is used by Tadeusz Banachiewicz in the 1930s because of the need to solve linear equations [2].

In the multiplication operation defined by the well-known Arthur Cayley, the order of the elements is taken into account. Many people build logic in this order. The first letter is written first and then the following sound symbols. A meaningful expression is formed. Cayley's matrix multiplication is known as obtaining $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$, which is formed by writing matrices $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times r}$ as $AB = C = [c_{ik}]_{m\times r}$, respectively taking rows from matrix A and columns from matrix B. The result of this binary operation gives a single matrix.

There is no commutative property for the left product. If commutative structures are to be obtained, the right product is needed.

Let $\mathbb{F}$ be a field. The set of all n– by –n matrices over a field $\mathbb{F}$ is denoted by $\mathbb{K}_n(\mathbb{F})$. That means

$$\mathbb{K}_n(\mathbb{F}) = \left\{[a_{ij}]_n | a_{ij} \in \mathbb{F}\right\}.$$

The set of all regular matrices order n over a field $\mathbb{F}$ is denoted by

$$\mathbb{M}_n(\mathbb{F}) = \left\{[a_{ij}]_n | a_{ij} \in \mathbb{F}\right\}.$$

The transpose of a matrix $A \in \mathbb{K}_n(\mathbb{F})$ is denoted by A^T.

2. Main results

A matrix A is said to be similar to matrix B if there exists a regular matrix P such that $A = P^{-1}BP$. There is not only one matrix B that is similar to the matrix A. Any regular matrix has infinitely many similar matrices. These two similar matrices are not commutativity too. The definition of right multiplication is the necessity for this. The definitions of left and right product are given below.

Definition 1. *Suppose that two matrices $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times r}$ and the product C of matrix A with matrix B is*

$$AB = C = [c_{ik}]_{m\times r}, \text{ where } c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

Definition 2. *Let $A = [a_{ij}]_{r\times m}$, $B = [b_{jk}]_{n\times r}$. Matrix $C = [\sum_{i=1}^{r} b_{ki}a_{ji}]_{n\times m}$ is called the right product of matrix A and matrix B. It is denoted by*

$$\underleftarrow{AB} = C = [c_{kj}]_{n\times m}, \text{ where } c_{kj} = \sum_{i=1}^{r} b_{ki}a_{ji}.$$

Example 1. *For two matrices $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\underleftarrow{BA}$ is*

$$\underleftarrow{AB} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 1 \\ 4 & 6 & 1 \end{bmatrix}.$$

$\underleftarrow{BA}$ cannot be made.

$$\text{If } A = [a_{1j}]_{1\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \end{bmatrix}, B = [b_{n1}]_{n\times 1} = \begin{bmatrix} b_{11} \\ \cdot \\ \cdot \\ \cdot \\ b_{n1} \end{bmatrix} \text{ then}$$

$$AB = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1}b \end{bmatrix} = \underleftarrow{BA}.$$

The commutativity property is satisfied for matrices of given structures.

We are given below the definition of the right product which is compatible with definition Cayley's.

Lemma 1. *Let $A, B \in \mathbb{K}_n(\mathbb{F})$. The following equality holds.*

$$\left((BA)^T\right)^T = BA.$$

The product of the reciprocal columns of two matrices with the same columns is defined as the Cracovian product. If A, B are any two square matrices of order n then the Cracovian product is

$$A \wedge B = B^T A, \text{ where } B^T \text{ is transpose of matrix } B \text{ in [4].}$$

Example 2. *For matrices in example 1 , we have*

$$B \wedge A = A^T B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 1 & 1 \end{bmatrix}.$$

Lemma 2. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*
$$(A \wedge B^T)^T = B^T \wedge A.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then
$$(A \wedge B^T)^T = (BA)^T = A^T B^T = B^T \wedge A.$$
□

Lemma 3. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*
$$(A^T \wedge B)^T = B \wedge A^T.$$

Proof. If $A, B \in \mathbb{M}_n(\mathbb{F})$, then
$$(A^T \wedge B)^T = (B^T A^T)^T = AB = B \wedge A^T.$$
□

Lemma 4. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*
$$(A \wedge B)^T = B \wedge A.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then
$$(A \wedge B)^T = (B^T A)^T = A^T B = B \wedge A.$$
□

Corollary 1. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*
$$(B \wedge A)^T = A \wedge B.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then
$$(B \wedge A)^T = (A^T B)^T = B^T A = A \wedge B.$$
□

Corollary 2. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*
$$(A^T \wedge B^T)^T = B^T \wedge A^T.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then
$$(A^T \wedge B^T)^T = (BA^T)^T = AB^T = B^T \wedge A^T.$$
□

Definition 3. *Let $A, B \in \mathbb{M}_n(\mathbb{F})$. The determinant of the new matrix obtained by writing the i^{th} column of the matrix B on the j^{th} column of the matrix A is called the co-divisor by column of the matrix B by the column on the matrix A. It is denoted by $\begin{pmatrix} B_{i_j} \\ A \end{pmatrix}$. Their number is n^2. The matrix co-divisor by columns is $\left[\begin{pmatrix} B_{i_j} \\ A \end{pmatrix}_{ij}\right]$ in [6].*
For the two matrices satisfying the above conditions, the matrix division is also given by
$$\frac{B}{A} := \frac{1}{|A|}\left[\begin{pmatrix} B_{i_j} \\ A \end{pmatrix}_{ji}\right], \tag{1}$$
and at the same time, the solution of the equation $AX = B$ is $X = \dfrac{B}{A}$ in [6], [7].

The transpose of $A \in \mathbb{M}_n(\mathbb{F})$ is denoted by A^T.

Definition 4. *Let A and B be two regular square matrices of order n. The determinant of the new matrix obtained by writing the i^{th} row of the matrix A on the j^{th} row of the matrix B is called the co-divisor by row of the matrix A by the row on the matrix B. It is denoted by* $\begin{pmatrix} AB \\ ij \end{pmatrix}$. *Their number is n^2. The matrix co-divisor by row is* $\left[\begin{pmatrix} AB \\ ij \end{pmatrix}_{ij}\right]$ *in [6].*

Let us the following theorem.

Theorem 1. *[3] Let $A, B \in \mathbb{M}_n(\mathbb{F})$, then the solution of the linear matrix equation $XA = B$ is*

$$X = \left(\frac{B^T}{A^T}\right)^T \text{.(see defrinition 3).}$$

Proof. The solution of the equation $AX = B$ is $X = \dfrac{B}{A}$ for all $A, B, X \in \mathbb{M}_n(\mathbb{F})$.

$$XA = B \iff (XA)^T = B^T \iff A^T X^T = B^T. \tag{2}$$

$$X^T = \frac{1}{|A^T|}\left[\begin{pmatrix} B^T A^T \\ ij \end{pmatrix}_{ij}\right] \Longrightarrow X = \frac{1}{|A^T|}\left[\begin{pmatrix} B^T A^T \\ ij \end{pmatrix}_{ji}\right]^T, \tag{3}$$

$$X = \frac{1}{|A^T|}\left[\begin{pmatrix} B^T A^T \\ ij \end{pmatrix}_{ij}\right] = \left(\frac{B^T}{A^T}\right)^T. \tag{4}$$

□

3. The Right Multiplication of Matrices

In this section, comparisons between right products of matrices and some known products are given. We are used square matrices.

Theorem 2. *For two matrices $A, B \in \mathbb{K}_n(\mathbb{F})$, we have the equality:*

$$\underleftarrow{AB} = BA.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then $A = [a_{ij}]_n$ and $B = [b_{jk}]_n$, then $A^T = [a_{ji}]_n$, $B^T = [b_{kj}]_n$ and

$$\left((BA)^T\right)^T = BA,$$

by definition 2 and lemma 1 it is deduced that:

$$\underleftarrow{AB} = BA.$$

□

The orders of the matrices are considered to be suitable for multiplication in the proof of the above theorem 2.

The following theorem is stated which provides the intersection with the commutativity property.

Corollary 3. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then the following equation holds.*

$$AB = \underleftarrow{BA}.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then by theorem 2

$$AB = \underleftarrow{BA}.$$

Because the product of the rows of matrix A and the columns of matrix B is AB. The product of $(B^T A^T)^T$ is explained the same description too. This product is equal to the right product of $\underleftarrow{BA}$. □

Example 3. *For two matrices* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, *AB right product is*

$$\underleftarrow{BA} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 10 & 7 \end{bmatrix}$$

and

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 10 & 7 \end{bmatrix}.$$

The following equality is achieved.

$$\underleftarrow{BA} = \begin{bmatrix} 4 & 3 \\ 10 & 7 \end{bmatrix} = AB$$

The property similar to the associative property in matrix multiplication is given below.

Proposition 1. *If $A, B, C \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{A(BC)} = \underleftarrow{(AB)C}.$$

Proof. Suppose $A, B, C \in \mathbb{K}_n(\mathbb{F})$,

$$\underleftarrow{A(BC)} = \underleftarrow{(BC)A} = (CB)A = C(BA) = C\underleftarrow{(AB)} = \underleftarrow{(AB)C}. \tag{5}$$

□

One of the few properties of multiplication of real numbers that generalize to matrices is the distribution property.

Proposition 2. *If $A, B, C \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{A\,(B + C)} = \underleftarrow{AB} + \underleftarrow{AC}.$$

Proof. Suppose $A, B, C \in \mathbb{K}_n(\mathbb{F})$,

$$\underleftarrow{A\,(B + C)} = (B + C)\,A = BA + CA = \underleftarrow{AB} + \underleftarrow{AC}.$$

□

Lemma 5. *If $A, B \in \mathbb{M}_n(\mathbb{F})$, then*

(i) $A = (\frac{(\underleftarrow{BA})^T}{B^T})^T$.

(ii) $B = \dfrac{\underleftarrow{BA}}{A}$.

Proof. The proof of the lemma is clear by theorem 1 and corollary 3. □

Proposition 3. *If $A \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{\overbrace{A \dots A}^{s\text{-times}}} = A^s, \; \textit{where } s \in \mathbb{Z}^+.$$

Proposition 4. *Suppose $A, B \in \mathbb{M}_n(\mathbb{F})$, If $\underleftarrow{BA} = I_n$, $\underleftarrow{AB} = I_n$ then $B = A^{-1}$.*

Proof. If $\underleftarrow{BA} = I_n$ and $\underleftarrow{AB} = I_n$. By corollary 3

$$\underleftarrow{BA} = AB = I_n, \; \underleftarrow{AB} = BA = I_n$$

then $B = A^{-1}$. □

The inverse of the right product is similar to that of the matrix product. It is as that.

$$\underleftarrow{(AB)}^{-1} = (BA)^{-1} = A^{-1}B^{-1} = \underleftarrow{B^{-1}A^{-1}}.$$

Similar situation is also ensured in transpose. That means:

$$\underleftarrow{(AB)}^T = (BA)^T = A^T B^T = \underleftarrow{B^T A^T}.$$

The relation between the right matrix product and the Cracovian product is given by the following lemma.

Lemma 6. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$A \wedge B = \underleftarrow{AB^T}.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$ and by theorem 2 , then

$$A \wedge B = (B)^T A = \underleftarrow{AB^T}.$$

□

Lemma 7. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$B^T \wedge A^T = \underleftarrow{B^T A}.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$ and by corollary 3 , then

$$B^T \wedge A^T = AB^T = \underleftarrow{B^T A}.$$

□

Lemma 8. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$A \wedge B^T = \underleftarrow{AB}$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$ and by corollary 3 , then

$$A \wedge B^T = AB = \underleftarrow{AB}.$$

□

Corollary 4. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{AB^T} = A \wedge B.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$ then

$$\underleftarrow{AB^T} = B^T A = A \wedge B.$$

□

Lemma 9. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$\left(A^T \wedge B\right)^T = \underleftarrow{BA}.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then

$$\left(A^T \wedge B\right)^T = \left(B^T A^T\right)^T = AB = \underleftarrow{BA}.$$

□

Theorem 3. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$\left(A^T \wedge B^T\right)^T = \left(\underleftarrow{A^T B}\right)^T.$$

Proof. If $A, B \in \mathbb{K}_n(\mathbb{F})$, then

$$\left(A^T \wedge B^T\right)^T = \left(BA^T\right)^T = AB^T,$$

$$\left(\underleftarrow{A^T B}\right)^T = \left(BA^T\right)^T = AB^T.$$

By equation 3 and equation 3 the following equality is obtained:

$$\left(A^T \wedge B^T\right)^T = \left(\underleftarrow{A^T B}\right)^T.$$

□

Corollary 5. *If $A, B \in \mathbb{K}_n(\mathbb{F})$, then*

$$\underleftarrow{A^T B} = A^T \wedge B^T.$$

4. Conclusion

Some results between the right matrix product, the matrix product and the Cracovian product are presented in this study. Multiplication of matrices is subjected to directional processing. The right multiplication is likely to contribute to applications in theoretical physics. This study is created an intersection for the Cracovian product, product and right product. Similar matrices are open problems for this operation. This multiplication is opened fields of application for the applied sciences of the matrix.

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